ISRAEL JOURNAL OF MATHEMATICS 83 (1993), 275–287

THE EXISTENCE OF INVARIANT σ-FINITE MEASURES FOR A GROUP OF TRANSFORMATIONS

BY

PIOTR ZAKRZEWSKI*

Institute of Mathematics, University of Warsaw
ul. Banacha 2, 00-913 Warsaw 59, Poland
and
Fachbereich Mathematik, TU Berlin
Strasse des 17 Juni 136, D-1000 Berlin 12, Germany

ABSTRACT

Suppose G is a group of measurable transformations of a σ-finite measure space $(X, \mathcal{A}, m)$. The main result of this paper gives necessary and sufficient conditions for the existence of a G-invariant, σ-finite measure defined on $\mathcal{A}$ and dominating the measure m in the sense of absolute continuity.

An example is also given of a σ-finite nonatomic measure space $(X, \mathcal{A}, m)$ together with a countable group G of its measurable transformations such that no G-invariant, σ-finite nonatomic measure exists on $\mathcal{A}$. Whether the Lebesgue space $([0,1], \mathcal{L}, \lambda)$ provides such an example, depends on set-theoretic assumptions.

1. Introduction

This paper may be viewed as a continuation of [15], where the problem of existence of invariant, *probability* measures was discussed. In the present paper the attention is shifted to *σ-finite* measures.

As pointed out in [15], we approach the problem from two different but closely related points of view.

(A) Motivated by ergodic theory one tries, given a group G of measurable transformations of a σ-finite measure space $(X, \mathcal{A}, m)$, to find necessary and

* This paper was written while the author was visiting the Technische Universitat Berlin as a research fellow of the Alexander von Humboldt Foundation.
Received December 31, 1991

sufficient conditions for the existence of a G-invariant measure μ defined on $\mathcal{A}$ and dominating m in the sense of absolute continuity (see [5, p. 81]).

(B) Influenced by Tarski's work on finitely additive invariant measures one searches for "purely combinatorial" conditions for the existence of an arbitrary G-invariant measure, when only a group G of bijections of a set X and a G-invariant σ-algebra $\mathcal{A}$ of subsets of X are given (see [13, p. 136]).

However, for reasons explained in [15], the two approaches lead to closely related results.

The prototype of such results is a classical theorem by Hopf [7]. For a cyclic group G of measurable transformations of a measure space $(X, \mathcal{A}, m)$ with a G-quasi-invariant, σ-finite measure m, Hopf formulated the notion of boundedness of a set and showed that the condition that X is bounded is necessary and sufficient for the existence of a G-invariant, probability measure equivalent with m.

Kawada [8], Hajian and Ito [4], and Chuaqui [2] generalized Hopf's theorem to arbitrary groups of measurable transformations of $(X, \mathcal{A}, m)$, and the author freed their results from the assumption that m is G-quasi-invariant [15, Theorem 3.1].

Kawada [8] found another generalization of Hopf's theorem. He proved that for an arbitrary group G of measurable transformations of a measure space $(X, \mathcal{A}, m)$ with a G-quasi-invariant, σ-finite measure m, the condition that X is σ-bounded, i.e. it is the countable union of measurable sets, bounded in the sense of Hopf, is necessary and sufficient for the existence of a G-invariant, σ-finite measure equivalent with m. This result, for the special case of a cyclic G, was later rediscovered by Halmos [6] who also noticed that in this case the assumption that the measure m is G-quasi-invariant may be omitted if we want the resulting invariant measure only to dominate m.

The main result of this paper generalizes the results of Kawada and Halmos to arbitrary groups of measurable transformations of any σ-finite measure space $(X, \mathcal{A}, m)$ (Theorem 3.1). The new element that arises here is the necessity of adding to the condition that X is σ-bounded another property, which guarantees the existence of a G-quasi-invariant measure dominating m (and hence is superfluous if m is assumed to be G-quasi-invariant). As a corollary a generalization of a related result by Arnold [1] is obtained (Theorem 3.3).

Next we consider the special case when G is countable and discuss the question of existence of an arbitrary, i.e. not necessary related to m, G-invariant, σ-finite measure on $\mathcal{A}$. An example is given of a nonatomic measure space whose underlying σ-algebra carries no such (nonatomic) measures for a certain countable group G of its measurable transformations (Proposition 4.2). The problem, whether just the ordinary Lebesgue space $([0,1], \mathcal{L}, \lambda)$ could serve as such an example, turns out, however, to depend on additional set-theoretic assumptions (Proposition 4.4). On the other hand, if we consider only the σ-algebra $\mathcal{B}$ of Borel subsets of [0,1], then for every countable group of Borel automorphisms of [0,1] an invariant, σ-finite measure always exists on $\mathcal{B}$ (Proposition 4.3).

Kawada [8] is not quoted in any of important later papers [6], [4], [2], [1] on the subject. I apologize for not mentioning his name in the earlier version [16] of this work and for mistakenly attributing one of his results to myself (see [16, Theorem 3.1.B]).

2. Definitions and preliminaries

We keep the notation and definitions of [15].

For the rest of this section let G be a group of measurable transformations of a σ-finite measure space $(X, \mathcal{A}, m)$.

A set $Z \in \mathcal{A}$ is called G-bounded with respect to m (or, simply, bounded, if G and m are clear from the context), if $Z \sim_\infty A$ implies $m(Z \setminus A) = 0$ for every $A \in \mathcal{A}$, $A \subseteq Z$, i.e. Z is not countably G-equidecomposable in $\mathcal{A}$ with any measure-theoretically proper measurable subset of itself (see [7]).

It is not difficult to prove that Z is bounded iff the measure m vanishes on all measurable subsets E of Z with the property that Z contains pairwise disjoint sets $E_n \in \mathcal{A}$, $n \in \mathbb{N}$, with each $E_n \sim_\infty E$ (see [8, Lemma 5]. Hence, in the terminology of [15], X is bounded iff m vanishes on all G-negligible measurable sets.

We shall later use the following local version of Theorem 3.1 from [15].

Proposition 2.1: *For every set $Z \in \mathcal{A}$ of positive measure the following conditions are equivalent:*

(i) *Z is G-bounded with respect to m.*

(ii) *There exists a probability measure μ on $\mathcal{A}$ such that μ is G-invariant on Z, $\mu(Z) = 1$ and for every $E \subseteq Z$, $m(E) = 0$ whenever $\mu(E) = 0$.*

If, additionally, m is G-quasi-invariant on Z, then (i) *is equivalent to:*

(iii) *There exists a probability measure μ on $\mathcal{A}$ such that μ is G-invariant on Z, $\mu(Z) = 1$ and for every $E \subseteq Z$, $m(E) = 0$ iff $\mu(E) = 0$.*

Proof: Clearly, (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i).

To prove the converse implications, find a group $\bar{G}$ of measurable transformations of the measure space $(Z, \mathcal{A} \cap \mathcal{P}(Z), m \mid \mathcal{A} \cap \mathcal{P}(Z))$, where $\mathcal{P}(Z)$ denotes the power set of Z, with the following property: if A and B are measurable subsets of Z, and $\bar{g}A = B$ for a certain $\bar{g} \in \bar{G}$, then A is countably G-equidecomposable with B in $\mathcal{A}$ (see [14, the proof of Theorem 2.6]).

Note that Z is $\bar{G}$-bounded with respect to $m \mid \mathcal{A} \cap \mathcal{P}(Z)$. Hence by [15, Theorem 3.1], there exists a $\bar{G}$-invariant, probability measure $\bar{\mu}$ on $\mathcal{A} \cap \mathcal{P}(Z)$ such that for every $A \in \mathcal{A} \cap \mathcal{P}(Z)$, $\bar{\mu}(A) = 0$ iff $m(\bar{g}A) = 0$ for every $\bar{g} \in \bar{G}$.

Finally, define μ by:

$$\mu(A) = \bar{\mu}(A \cap Z) \quad \text{for } A \in \mathcal{A}.$$

The measure μ has all the required properties. ∎

We say that X is $\sigma - G$-bounded with respect to m (or, simply, σ-bounded), if X is the countable union of measurable bounded sets.

Kawada [8, Satz 2] proved that if the measure m is G-quasi-invariant, then the condition that X is $\sigma - G$-bounded is necessary and sufficient for the existence of a G-invariant, σ-finite measure on $\mathcal{A}$ equivalent with m.

The following is an extended version of Lemma 3.2 from [15].

PROPOSITION 2.2: *If m is G-quasi-invariant (resp. G-invariant) on a set Z of positive measure and the σ-ideal $I_G(Z)$ is σ-saturated in $\mathcal{A}$, then there exists a G-quasi-invariant (resp. G-invariant), σ-finite measure ν on $\mathcal{A}$ such that $I_\nu = I_G(Z)$ and $\nu(X \backslash Z^*) = 0$, where $Z^* = \bigcup_n g_n Z$ for certain functions g_n from G.*

Proof: Let $\mathcal{K} = \{D_n\}$ be a maximal family of pairwise disjoint, measurable sets with associated functions $h_n \in G$ such that $h_n D_n \subseteq Z$ and $0 < m(h_n D_n) < \infty$ for every n.

Define ν by:

$$\nu(A) = \sum_n m(h_n[A \cap D_n]) \quad \text{for } A \in \mathcal{A}.$$

Clearly, ν is a measure on $\mathcal{A}$ and:

$$(1) \qquad \nu(X \backslash \bigcup_n D_n) = 0.$$

Hence ν is σ-finite and it suffices to let $g_n = h_n^{-1}$ for each n to have $\nu(X\backslash Z^*) = 0$ for $Z^* = \bigcup_n g_n Z$.

Since it was already proved in [15] that if m is G-quasi-invariant on Z, then ν is G-quasi-invariant and $I_\nu = I_G(Z)$, the only new detail is to show that the G-invariance of m on Z implies the G-invariance of ν.

So assume that m is G-invariant on Z.

To prove that ν is G-invariant, take arbitrary $A \in \mathcal{A}$ and $g \in G$, and consider first the special case in which $A \subseteq D_k$ and $gA \subseteq D_n$ for certain k and n. Then, by the definition of ν and the G-invariance of m on Z,

$$\nu(gA) = m(h_n gA) = m(h_k g^{-1} h_n^{-1} h_n gA) = m(h_k A) = \nu(A).$$

For the general case, use (1), the G-invariance of ν and the above to justify the following chain of equalities:

$$\begin{aligned}\nu(gA) &= \sum_n \nu(gA \cap D_n) = \sum_n \sum_k \nu(gA \cap D_n \cap gD_k) \\ &= \sum_n \sum_k \nu(g[A \cap g^{-1} D_n \cap D_k]) = \sum_n \sum_k \nu(A \cap g^{-1} D_n \cap D_k) \\ &= \sum_n \nu(A \cap g^{-1} D_n) = \nu(A).\end{aligned}$$

Thus the measure ν has all the required properties. ∎

3. The existence of an invariant, σ-finite measure dominating a given measure

In this section we prove the main result of this paper which is formulated as follows.

THEOREM 3.1: *Let G be a group of measurable transformations of a σ-finite measure space $(X, \mathcal{A}, m)$.*

Then the following conditions are equivalent:

(i) *There exists a G-invariant, σ-finite measure μ on $\mathcal{A}$ such that $m \ll \mu$.*

(ii) *X is σ-G-bounded with respect to m and the σ-ideal $I_G(X)$ is σ-saturated in $\mathcal{A}$.*

Proof: We first prove that (i)⇒(ii).

To see that X is σ-bounded with respect to m, consider a family $\{X_n: n \in \mathbb{N}\}$ of measurable sets of finite μ-measures such that $X = \bigcup_{n\in\mathbb{N}} X_n$. Note that since $m \ll \mu$, each X_n being bounded with respect to μ is also bounded with respect to m.

The second assertion of (ii) follows from the inclusion $I_\mu \subseteq I_G(X)$.

To prove that (ii)$\Rightarrow$(i) it suffices to establish the following.

CLAIM: *Condition* (ii) *implies that $I_G(X)$ is the σ-ideal of a G-invariant, σ-finite measure μ on $\mathcal{A}$.*

Proof of the claim: The construction of μ is based on the following lemma which can be easily established by combining Propositions 2.1 and 2.2.

LEMMA 3.2: *If m is G-quasi-invariant on a bounded set Z of positive measure and the σ-ideal $I_G(Z)$ is σ-saturated in $\mathcal{A}$, then there exists a G-invariant, σ-finite, measure ν on $\mathcal{A}$ such that $I_\nu = I_G(Z)$ and $\nu(X\setminus Z^*) = 0$, where $Z^* = \bigcup_n g_n Z$ for certain functions g_n from G.*

Assume now that X is σ-bounded and the σ-ideal $I_G(X)$ is σ-saturated in $\mathcal{A}$.

Note that $I_G(X) \subseteq I_m$. So, by [15, Proposition 2.1], there exists a set $Y \in \mathcal{A}$ such that $m(X\setminus Y) = 0$ and m is G-quasi-invariant on Y. Since X is σ-bounded, $Y = \bigcup_{n\in\mathbb{N}} A_n$ for certain bounded sets $A_n \in \mathcal{A}, n \in \mathbb{N}$.

Define by induction a new sequence $< Z_n >$ of bounded subsets of Y of positive measures, associating with each Z_n a G-invariant, σ-finite measure ν_n on $\mathcal{A}$ and a set Z_n^* by applying Lemma 3.2 to $Z = Z_n$.

Let $n_0 = \min\{n: m(A_n) > 0\}$ and set $Z_0 = A_{n_0}$.

Assume that $k > 0$ and Z_i, ν_i, and Z_i^* are already defined for $i < k$.

If $m(X\setminus \bigcup_{i<k} Z_i^*) > 0$, let $n_k = \min\{n: m(A_n\setminus \bigcup_{i<k} Z_i^*) > 0\}$ and set $Z_n = A_{n_k}\setminus \bigcup_{i<k} Z_i^*$.

If $m(X\setminus \bigcup_{i<k} Z_i^*) = 0$, the procedure stops after only k steps.

Finally, set $\mu = \sum_n \nu_n$.

Clearly, μ is a G-invariant measure on $\mathcal{A}$.

It remains to check that μ is σ-finite and $I_\mu = I_G(X)$.

To prove the first assertion, notice that if $i < k$, then $\nu_i(Z_k^*) = 0$. For if not, then since $I_{\nu_i} = I_G(Z_i)$ there is $g \in G$ such that $m(Z_i \cap gZ_k^*) > 0$. But since $Z_k^* = \bigcup_n g_n Z_k$ for certain countably many functions g_n from G, this in turn implies that there is $g' \in G$ with $m(Z_i \cap g'Z_k) > 0$. Hence $\nu_i(Z_k) > 0$, contrary to the fact that $Z_k \cap Z_i^* = \emptyset$ and $\nu_i(X\setminus Z_i^*) = 0$.

It follows that μ, being the sum of countably many mutually orthogonal, σ-finite measures, is itself σ-finite.

For the second assertion, note first that $m(X\setminus\bigcup_n Z_n^*) = 0$, so $I_G(X) = \bigcap_n I_G(Z_n^*)$. But since m is G-quasi-invariant on Y and $m(X\setminus Y) = 0$, $I_G(Z_n^*) = I_G(Z_n)$ for each n. It follows that $I_G(X) = \bigcap_n I_{\nu_n} = I_\mu$.

This completes the proof of the claim and of the theorem. ∎

It is worth noting that by the above proof, if, under the hypotheses of Theorem 3.1, the set of all G-invariant, σ-finite measures on $\mathcal{A}$ dominating the measure m is non-empty, then it has the least element in the sense of the partial (pre-)ordering $\ll$.

Let us briefly discuss the question, whether both assertions of condition (ii) of the above theorem are essential.

In order to see that the implication "$I_G(X)$ is σ-saturated $\Rightarrow$ X is σ-bounded" is false, consider any of the well known examples from ergodic theory in which there is no G-invariant, σ-finite measure equivalent with a given G-quasi-invariant measure m (a simple one is: $X =$ the real line, $\mathcal{A} =$ the σ-algebra of Borel subsets of $X, m =$ the Lebesgue measure on $\mathcal{A}$, $G =$ the group of similarities of the form $x \to \rho x + \sigma$, where ρ and σ are rationals, $\rho \neq 0$—see [9, Example (α), p.159]).

The converse implication is also easily seen to be false. Just consider X and $\mathcal{A}$ as above, $m =$ the $\{0,1\}$-valued measure concentrated on a point and $G =$ the group of all translations. It is worth comparing this with the fact that the boundedness of X already implies that $I_G(X)$ is σ-saturated (see [15, Lemma 3.4]).

Let us also recall that the condition "$I_G(X)$ is σ-saturated" is equivalent to each of the following assertions (see [15, Lemma 3.3]):

(i) There exists a G-quasi-invariant, σ-finite measure μ on $\mathcal{A}$ such that $I_\mu = I_G(X)$.

(ii) There exists a G-quasi-invariant, σ-finite measure μ on $\mathcal{A}$ such that $m \ll \mu$.

Remark: The above equivalences suggest another method of proving the implication (ii)$\Rightarrow$ (i) of Theorem 3.1: take a measure μ on $\mathcal{A}$ with $I_\mu = I_G(X)$, show that X is σ-G-bounded with respect to μ and then apply Kawada's result. The proof presented in the paper seems, however, to provide a better insight to the problem.

We conclude this section with the following corollary of Theorem 3.1, which generalizes a theorem by Arnold [1, Theorem 1], giving another necessary and sufficient condition for the existence of an invariant, σ-finite measure.

THEOREM 3.3: *Under the hypotheses of Theorem 3.1 the following conditions are equivalent:*

(i) *There exists a G-invariant, σ-finite measure μ on $\mathcal{A}$ such that $m \ll \mu$.*

(ii) *The σ-ideal $I_G(X)$ is σ-saturated in $\mathcal{A}$ and for each $\epsilon > 0$ X is, up to a m-null set, the union of countably many pairwise disjoint measurable sets X_n such that for every $n \in \mathbb{N}$ the following condition is satisfied: for every $g \in G$ and $A \in \mathcal{A}$, if $A \subseteq X_n$ and $gA \subseteq X_n$, then*

$$m(A)/(1+\epsilon) \leq m(gA) \leq (1+\epsilon)m(A).$$

(iii) *The σ-ideal $I_G(X)$ is σ-saturated in $\mathcal{A}$ and X is, up to a m-null set, the union of countably many pairwise disjoint measurable sets X_n, $n \in \mathbb{N}$, such that for every $n \in \mathbb{N}$ the following condition is satisfied: there is $k_n > 0$ such that for every $g \in G$ and $A \in \mathcal{A}$, if $A \subseteq X_n$ and $gA \subseteq X_n$, then $m(gA) \geq k_n \cdot m(A)$.*

Proof: To prove that (i)$\Rightarrow$ (ii), first find $Y \in \mathcal{A}$ such that $m(X \backslash Y) = 0$ and m is G-quasi-invariant on Y. Then follow the proof of Theorem 1 from [1] to obtain a required decomposition of Y.

The implication (ii)$\Rightarrow$ (iii) is obvious.

To prove that (iii)$\Rightarrow$ (i), note that under the hypotheses of (iii) the sets X_n, $n \in \mathbb{N}$, and the set $X \backslash \bigcup_{n\in\mathbb{N}} X_n$ are bounded. So, X is σ-bounded and everything follows now from Theorem 3.1. ∎

4. The existence of an arbitrary invariant, σ-finite measure

Our next corollary of Theorem 3.1 generalizes Theorem 3.5 from [15] giving a solution to the problem of finding necessary and sufficient conditions for the existence of an arbitrary G-invariant, σ-finite measure on $\mathcal{A}$, if $\mathcal{A}$ is only assumed to be a G-invariant σ-algebra of subsets of X.

THEOREM 4.1: *Let G be a group of bijections of a set X and $\mathcal{A}$ a G-invariant σ-algebra of subsets of X.*

Then the following conditions are equivalent:

(i) *There exists a G-invariant, σ-finite measure on $\mathcal{A}$.*

(ii) *There exists a σ-finite measure m on $\mathcal{A}$ and a set $A \in \mathcal{A}$ of positive m-measure such that A is bounded and the σ-ideal $I_G(A)$ is σ-saturated in $\mathcal{A}$.*

The usefulness of the above theorem for proving the existence of invariant measures is doubtful. However, in particular cases in which one already knows that for certain other reasons a G-invariant, σ-finite measure on $\mathcal{A}$ does not exist, it gives interesting information about invariance (or rather: non-invariance) properties of σ-finite measures on $\mathcal{A}$.

To illustrate this point of view, let us consider the case when the group G is countable.

Is it then possible that $\mathcal{A}$ carries no G-invariant, σ-finite measure at all ?

We have to make two restrictions in order to avoid trivial answers.

First, if A is an atom of the σ-algebra $\mathcal{A}$ (i.e. $A \neq \emptyset$ and $A' \in \mathcal{A}$, $A' \subseteq A$ imply that either $A' = \emptyset$ or $A' = A$), then one immediately defines a G-invariant, σ-finite measure on A concentrated on the set $\bigcup_{g \in G} gA$. Hence we restrict ourselves to measures which vanish on all atoms of $\mathcal{A}$. From now on we assume, moreover, that $\mathcal{A}$ contains all singletons. In this case a measure which vanishes on all atoms of $\mathcal{A}$ (= the singletons of X) is called nonatomic.

Secondly, certain σ-algebras carry no σ-finite, nonatomic measures at all, even without the additional requirement of G-invariance. Hence we also assume that the set $\mathcal{M}$ of all σ-finite, nonatomic measures on $\mathcal{A}$ is non-empty.

Now we ask, whether it is possible that every measure $m \in \mathcal{M}$ is *not* G-invariant. Note that if this is the case, then by Theorem 4.1, the group G must take care that for each $m \in \mathcal{M}$ every set A of positive measure is not G-bounded with respect to m (the countability of G implies that the σ-ideal $I_G(A)$ is σ-saturated in $\mathcal{A}$ — see [15, Proposition 2.2]).

It turns out that not only a countable but even a cyclic group can do the job.

To give an appropriate example it is necessary to recall some standard definitions and facts from ergodic theory.

We say that a G-quasi-invariant, σ-finite measure m on A is G-ergodic if $A \in \mathcal{A}$ satisfies $m(gA \backslash A) = 0$ for all $g \in G$ only if $m(A) = 0$ or $m(X \backslash A) = 0$. It is well known and easy to prove that if m_1 is another G-quasi-invariant, σ-finite measure on $\mathcal{A}$ with $m_1 \ll m$ and m is G-ergodic, then $m_1 \equiv m$.

Our construction is based on the classical result by Ornstein (see [3, Example 6.7, p. 83]), which states that there exists a cyclic group H of measurable transformations of the Lebesgue space $([0,1],\mathcal{L},\lambda)$ such that Lebesgue measure λ on the σ-algebra $\mathcal{L}$ of Lebesgue measurable subsets of [0,1] is H-quasi-invariant and H-ergodic but there is no H-invariant, σ-finite measure μ on $\mathcal{L}$ equivalent with λ.

THEOREM 4.2: *There exist a set X, a cyclic group G of bijections of X and a G-invariant σ-algebra $\mathcal{A}$ of subsets of X containing all singletons such that $\mathcal{A}$ carries σ-finite, nonatomic measures but none of them is G-invariant.*

Proof: Let Y be a subset of [0,1] such that $\lambda^*(Y) > 0$ but $\lambda^*(A) = 0$ for every $A \subseteq [0,1]$ of cardinality less than that of Y (λ^* denotes the Lebesgue outer measure).

Let H be the cyclic group of bijections of [0,1] given by Ornstein's result quoted above.

Now let:

$$X = \bigcup_{h\in H} hY, \quad G = \{h \mid X : h \in H\}, \quad \mathcal{A} = \{E \cap X : E \in \mathcal{L}\}.$$

Note that the H-ergodicity of λ implies that $\lambda^*(X) = 1$. Hence the formula:

$$m(E\cap X) = \lambda(E) \quad \text{for } E \in \mathcal{L}$$

properly defines a σ-finite, nonatomic measure on $\mathcal{A}$. Moreover, the measure m is G-quasi-invariant and G-ergodic.

Suppose now, towards a contradiction, that μ is a G-invariant, σ-finite, nonatomic measure on $\mathcal{A}$.

We claim that $\mu \ll m$.

Indeed, otherwise there is a λ-null set $E \in \mathcal{A}$ with $\mu(E) > 0$, so μ restricted to measurable subsets of E is a σ-finite, nonatomic measure defined on the power set of E. But, by a result of Kunen, if there exists a σ-finite, nonatomic measure defined on the power set of a set of cardinality κ, then there exists a subset of [0,1] of cardinality less than κ with positive outer Lebesgue measure (for a proof see [11, Theorem on page 478]). So, due to our cardinality assumption on Y, a contradiction is reached which proves the claim.

Now, the H-ergodicity of m implies that in fact $\mu \equiv m$.

Define a measure $\bar{\mu}$ on $\mathcal{L}$ by:

$$\bar{\mu}(A) = \mu(A\cap X) \quad \text{for } A \in \mathcal{L}.$$

Clearly, $\bar{\mu}$ is H-invariant and σ-finite. Moreover, $\mu \equiv m$ implies that $\bar{\mu} \equiv \lambda$. But this contradicts the choice of H, completing the proof that X, G and $\mathcal{A}$ have all required properties. ∎

It seems natural to ask whether we could find an example as above with simply $X = [0,1]$ and:
either

$$\mathcal{A} = \mathcal{B}, \quad \text{the } \sigma\text{-algebra of Borel subsets of } [0,1] \tag{1}$$

or

$$\mathcal{A} = \mathcal{L}. \tag{2}$$

It turns out that the answer is "no" in case (1) and, somewhat surprisingly, in case (2) it depends on our set-theoretic assumptions.

We shall use the result by Silver stating that if X is an uncountable Borel subset of [0,1] and G is a countable group of Borel automorphisms of X, then there exists an uncountable Borel set S which intersects every G-orbit (i.e. a set of the form $\{gx: g \in G\}$, $x \in X$) in at most one point (see [12, p.1]).

PROPOSITION 4.3: *If G is a countable group of Borel automorphisms of [0,1], then there exists a G-invariant, σ-finite, nonatomic measure on the σ-algebra $\mathcal{B}$ of Borel subsets of [0,1].*

Proof: Let S be the Borel set given by Silver's result quoted above. It is well-known (see [10, Theorem 8.1]) that there exists a probability, nonatomic measure ν on the σ-algebra of Borel subsets of S.

Define a measure μ on $\mathcal{B}$ by:

$$\mu(B) = \sum_{g \in G} \nu(S \cap gB) \quad \text{for } B \in \mathcal{B}.$$

Clearly, μ is G-invariant, σ-finite and nonatomic. ∎

PROPOSITION 4.4: *The following conditions are equivalent:*

(i) *For every countable group G of measurable transformations of the Lebesgue space $([0,1], \mathcal{L}, \lambda)$ there exists a G-invariant, σ-finite, nonatomic measure on $\mathcal{L}$.*

(ii) *There exists a probability, nonatomic measure defined on the power set of [0,1].*

Proof: The implication (i)⇒(ii) was essentially established in the course of proving Proposition 4.2. Indeed, the negation of (ii) implies that every σ-finite, nonatomic measure on $\mathcal{L}$ is absolutely continuous with respect to λ, so Ornstein's group contradicts (i).

To prove the converse, let G be an arbitrary group of measurable transformations of $([0,1],\mathcal{L},\lambda)$. Since each $g \in G$ is equal, modulo a λ-null set, to a Borel function and the group G is countable, there exists a Borel set $X \subseteq [0,1]$ such that $\lambda(X) = 1$ and $g \mid X$ is a Borel automorphism of X for every $g \in G$. By Silver's result, there is an uncountable Borel set S which intersects each G-orbit in at most one point.

Since the cardinality of S is the same as that of [0,1], condition (ii) implies that there exists a probability, nonatomic measure ν defined on the power set of S.

Define a measure μ on $\mathcal{L}$ by:

$$\mu(A) = \sum_{g \in G} \nu(S \cap gA) \quad \text{for } A \in \mathcal{L}.$$

Clearly, μ is G-invariant, σ-finite and nonatomic. ∎

It is well known that the status of the statement that there exists a probability, nonatomic (countably additive!) measure defined on the power set of [0,1] is most probably that of an additional set-theoretic axiom. It certainly cannot be proved in the usual set theory (say, ZFC) since its negation follows from the Continuum Hypothesis. On the other hand, in the common belief of specialists, it cannot also be in ZFC disproved and its interesting consequences are widely studied (see [11, p.468] and [13, p.138]).

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